

Yuri Safarov (1958–2015)



This issue is dedicated to Yuri Safarov who passed away on 2 June 2015. It is a collection of papers in subject areas related to Safarov's work which are written by people who knew him well.

The editor of this volume and author of this preface is a collaborator of Yuri Safarov. The two of us coauthored the book [16] which contains Safarov's most influential mathematical results. Lars Hörmander, who reviewed the book for the Bulletin of the LMS, summarised his review [3] as follows: "In the reviewer's opinion, this book is indispensable for serious students of spectral asymptotics".

Yuri was born in Leningrad in the Soviet Union on 23 January 1958. He spent his formative years in Leningrad (later renamed to Saint Petersburg), before moving to London in 1993. Yuri did his undergraduate studies at Leningrad State University, from which he graduated in 1980. Upon graduation, he was appointed Research Fellow at the Leningrad Branch of the Steklov Mathematical Institute.

Yuri did a PhD at the Steklov Institute under the supervision of Michael Solomyak. In his PhD thesis [12] he considered two problems.

Problem 1: spectral problem for the operator

$$\begin{pmatrix} 0 & -i \operatorname{curl} \\ i \operatorname{curl} & 0 \end{pmatrix} \quad (1)$$

acting on pairs $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ of solenoidal vector fields in a bounded domain $M \subset \mathbb{R}^3$ with smooth boundary ∂M . The boundary conditions are

$$u_2|_{\partial M} = 0, \quad u_1 \times n|_{\partial M} = 0, \quad (2)$$

where n is the normal to ∂M . Safarov succeeded in overcoming two fundamental difficulties evident in (1), (2): a) the difficulty that the problem is not elliptic and b) the difficulty that the problem is not semi-bounded (the eigenvalues λ_k accumulate at $+\infty$ and $-\infty$). Let us define the two counting functions $N_{\pm} : (0, +\infty) \rightarrow \mathbb{N} \cup \{0\}$ as $N_{\pm}(\lambda) := \sum_{0 < \pm \lambda_k < \lambda} 1$. The function $N_+(\lambda)$ counts eigenvalues λ_k between zero and λ , whereas the function $N_-(\lambda)$ counts eigenvalues λ_k between $-\lambda$ and zero. Safarov proved that

$$N_{\pm}(\lambda) = \frac{V}{3\pi^2} \lambda^3 + O(\lambda^2) \quad \text{as } \lambda \rightarrow +\infty,$$

where V is the volume of the domain M . Furthermore, under the nonperiodicity condition (not too many periodic billiard trajectories, see [16] for details) Safarov proved the stronger result

$$N_{\pm}(\lambda) = \frac{V}{3\pi^2} \lambda^3 + o(\lambda^2) \quad \text{as } \lambda \rightarrow +\infty.$$

Problem 2: transmission problem. Here we have two compact n -dimensional Riemannian manifolds, M_+ and M_- , connected by a common smooth boundary Γ and the aim is to study the spectrum of the Laplacian on the manifold $M := M_+ \cup M_-$ subject to natural transmission boundary conditions on the $(n-1)$ -dimensional hypersurface Γ . Here the fundamental difficulty, one which Safarov succeeded in overcoming, is that the billiard flow experiences branching on the hypersurface Γ : in addition to the usual reflected trajectory one gets a refracted trajectory. Under appropriate assumptions on the branching billiard flow (nonperiodicity plus the condition that there are not too many trajectories experiencing an infinite number of reflections in a finite time), Safarov obtained

a two-term asymptotic formula for the counting function of this problem. In fact, Safarov went further and examined the situation when there are many periodic trajectories in which case he derived an asymptotic formula with a periodic factor in the second term.

In 1990 the Steklov Institute awarded Yuri a DSc (habilitation). In the Soviet Union it was most unusual to be awarded a DSc at such a young age. Yuri's DSc thesis [13] contained a number of important results, some of which are listed below.

Firstly, Safarov performed a comprehensive analysis of the situation when there are many periodic trajectories. For a particular type of spectral problem such an analysis already started in [12]. However, in [13] it was done in far greater generality. Also, Safarov showed how to deal with the (very common) case when the periodic factor in the second asymptotic term of the counting function is discontinuous. Non-classical (i.e. not purely polynomial) two-term asymptotic formulae for the counting function are Yuri's trademark results.

Secondly, Safarov established very precise asymptotic formulae for the spectral function. The spectral function is a "local" version of the counting function: whereas the counting function depends only on one variable, the spectral parameter λ , the spectral function depends also on the point $y \in M$ or even a pair of points $x, y \in M$ (here M is the manifold on which the operator acts). For the case of a scalar positive definite operator the spectral function is defined below, see formula (4). Without going into technicalities let us just mention that establishing asymptotic formulae for the spectral function is a delicate matter and the geometric conditions involved in the derivation of such formulae are different from those for the "global" counting function. Asymptotic formulae for the spectral function are also Yuri's trademark results.

Thirdly, Safarov examined elliptic systems on manifolds without boundary. Let A be an elliptic (i.e. determinant of principal symbol is nonzero) self-adjoint first order linear pseudodifferential operator acting on m -columns of complex-valued half-densities over a connected closed (i.e. compact and without boundary) n -dimensional manifold M . We also assume that the eigenvalues of the principal symbol are simple, but we do not assume that the operator is semi-bounded. Under the above assumptions Safarov derived an explicit formula for the principal symbols of the m individual oscillatory integrals making up the propagator e^{-itA} , $t \in \mathbb{R}$ being the time variable. Safarov's formula was subsequently heavily used by a number of authors, myself included. The formula is nontrivial in that it differs fundamentally from the well known scalar formula of Duistermaat and Guillemin.

I met Yuri around 1983 and, despite the fact that we lived in different cities (Yuri in Leningrad and I in Moscow), we started interacting and exchanging ideas. Towards the end of the 1980s Simon Gindikin suggested that Yuri and I write a book for the American Mathematical Society. We took up the offer but our work progressed very slowly because we were preoccupied with difficulties of everyday life in precollapse USSR and because we did not have access to personal computers which would have allowed us to typeset the text in TeX. But things changed when the two of us moved to the UK: I took up an appointment at

Sussex in 1991 and Yuri one at King's College London in 1993. Yuri was initially appointed as an SERC Advanced Fellow. He became a Professor in 1997 and continued working at King's until his untimely death.

We finished writing our book [16] in 1996. The book deals with an elliptic self-adjoint scalar differential operator of even order $2m$ acting on half-densities over a compact n -dimensional manifold M with smooth boundary ∂M , subject to m boundary conditions on ∂M . Without loss of generality, the operator is assumed to be positive definite and the spectral parameter is denoted by λ^{2m} , which simplifies subsequent asymptotic formulae. The objects of study are the counting function

$$N(\lambda) := \sum_{\lambda_k < \lambda} 1 \quad (3)$$

and the spectral function

$$e(\lambda, x, y) := \sum_{\lambda_k < \lambda} v_k(x) \overline{v_k(y)}, \quad (4)$$

where the λ_k and the $v_k(x)$, $k = 1, 2, \dots$, are the eigenvalues and the orthonormalised eigenfunctions. Obviously, the counting function (3) is the trace of the spectral projection, whereas the spectral function (4) is the integral kernel of the spectral projection.

For the counting function $N(\lambda)$ the book establishes explicit two-term asymptotic formulae as $\lambda \rightarrow +\infty$. Here the second term may be classical (constant times λ^{n-1}) or non-classical (periodic function of λ times λ^{n-1}).

The spectral function $e(\lambda, x, y)$ is examined in the book on the diagonal and in the interior of the manifold, i.e. under the assumptions $x = y \notin \partial M$. It is shown that the classical second term in the asymptotics of $e(\lambda, y, y)$ is zero, whereas the non-classical one may be nonzero. Safarov provided a simple example of a situation when the second term in the asymptotics of $e(\lambda, y, y)$ is non-classical: this is the example of the Laplacian in a planar domain bounded by an ellipse, subject to the Dirichlet or Neumann boundary condition. When the point y is a focus of the ellipse we get a non-classical second term which Safarov wrote out explicitly.

The book uses a version of Levitan's hyperbolic equation method, so a propagator had to be constructed using microlocal techniques. Yuri insisted on using his particular microlocal technique based on an invariantly defined complex-valued phase function. This technique originates from the paper [8] and allows one to construct oscillatory integrals globally in time, passing without problems through caustics and avoiding the need for tackling compositions of oscillatory integrals. Furthermore, this approach leads to a new, purely analytic and straightforward, definition of the Maslov index.

In the latter part of his career, after finishing the book [16], Yuri studied a variety of different questions in the spectral theory of differential operators. Some of these publications are mentioned below.

In [7] Laptev and Safarov established Szegő type limit theorems for elliptic semi-bounded self-adjoint pseudodifferential operators.

In [14] Safarov developed a coordinate-free calculus of pseudodifferential operators on a manifold. Here the critical idea, originally due to Widom, is the use of a connection.

In [15] Safarov performed a comprehensive study of Fourier Tauberian theorems, coming up with an ‘ultimate’, very general version of a Fourier Tauberian theorem. He used it for the study of the counting function of the Dirichlet Laplacian in an arbitrary domain of finite volume.

In [11] Netrusov and Safarov studied the asymptotic distribution of eigenvalues of the Laplacian on a bounded domain in \mathbb{R}^n with rough boundary. The authors obtained an explicit remainder estimate in the Weyl formula for the Dirichlet Laplacian. Furthermore, they established sufficient conditions for the validity of the Weyl formula for the Neumann Laplacian.

In [6] Lapointe, Polterovich and Safarov studied the average growth of the spectral function on a Riemannian manifold. Two types of averaging were considered: with respect to the spectral parameter and with respect to a point on a manifold.

In [1] Filonov and Safarov studied, in the abstract setting, the following question: if the self-commutator $[A^*, A]$ of an operator A is small, is A close to a normal (or even a diagonal) operator?

In [10] Safarov, jointly with his PhD student McKeag, revisited the issue of coordinate-free description of pseudodifferential operators on manifolds. This paper was published at a time when I also became interested in pseudodifferential operators and Fourier integral operators on manifolds, and we had a series of fruitful discussions with Yuri in the last few years of his life. These discussions had a profound influence on my current research.

In [4] Jakobson, Safarov and Strohmaier studied the semiclassical limit of spectral theory on manifolds whose metrics have jump-like discontinuities and proved a quantum ergodicity theorem for such systems.

Safarov’s last paper [5], written jointly with his PhD student Kachkovskiy, was published posthumously. It revisits the self-commutator problem which was initially examined in [1]. The problem at hand is one posed by Halmos [2] and can be informally stated as follows: how close is an almost normal matrix to a normal one? Kachkovskiy’s and Safarov’s paper [5] effectively settles this long-standing problem.

Safarov’s contributions to mathematics were recognised by mathematical societies both in the USSR and the UK. The Leningrad Mathematical Society awarded Yuri the 1987 Young Mathematician Prize and The London Mathematical Society awarded him the 1996 Junior Whitehead Prize. The citation [9] for the latter read as follows.

Dr Y. Safarov of King’s College, London, is awarded a Junior Whitehead Prize for his work on the spectral analysis of partial differential operators and microlocal analysis.

He undertook in the late eighties a detailed investigation of compact manifolds for which the spectral asymptotics of the Laplace–Beltrami operator has a non-classical second term. The complete analysis of this involved looking at the geometrical structure of the set of periodic orbits and their relationship with

the behaviour of the wave equation — a physically-motivated approach which required completely new insights to make into a piece of rigorous mathematics.

Since this result, Dr Safarov has been engaged in several other major pieces of work, including a detailed asymptotic analysis of the spectral function, and a powerful abstract version of Szegő's theorem. He has also suggested a new approach to the theory of Fourier integral operators based on the use of a complex, rather than real, phase function. Amongst the advantages of this viewpoint is a new definition of the Maslov index.

Most recently, Safarov has developed a coordinate-free approach to the theory of pseudo-differential operators, introducing Weyl symbols for a pseudo-differential operator on a manifold with connection.

Throughout his career Yuri supervised many PhD students and several post-docs. He also volunteered to teach a course *Distributions, Fourier Transforms and Microlocal Analysis* at the London Taught Course Centre, an organisation which provides taught lecture courses for PhD students from several London universities and some universities from surrounding areas. He continued lecturing at the London Taught Course Centre when he was already ill. Yuri was an excellent lecturer at all university teaching levels, undergraduate and postgraduate.

Despite the progressing illness, Yuri actively supervised his last PhD student Paolo Battistotti and signed off the final version of the thesis just days before his death.

Outside of mathematics Yuri's passion was chess. As a youngster, participating in a tournament he played against the future world champion Garry Kasparov and that game ended in a draw. Even though Kasparov was five years younger than Yuri, this was a remarkable achievement. In 1981 Yuri was awarded the title Master of Sport of the USSR. He also played for the chess team of Leningrad. At some point in his life Yuri had to decide whether to pursue a professional chess career or become a professional mathematician. Fortunately for the subject of spectral theory he chose the latter.

Yuri was an open, friendly and outgoing person who enjoyed life to the fullest. He will be greatly missed by friends and colleagues.

Dmitri Vassiliev

References

- [1] N. Filonov and Yu. Safarov, On the relation between an operator and its self-commutator. *Journal of Functional Analysis* **260** (2011), 2902–2932.
- [2] P. R. Halmos, Some unsolved problems of unknown depth about operators on Hilbert space. *Proceedings of the Royal Society of Edinburgh: Section A* **76** (1976), 67–76.

- [3] L. Hörmander, The asymptotic distribution of eigenvalues of partial differential operators, book review. *Bulletin of the London Mathematical Society* **30** (1998), 205–209.
- [4] D. Jakobson, Yu. Safarov and A. Strohmaier, The semiclassical theory of discontinuous systems and ray-splitting billiards. *American Journal of Mathematics* **137** (2015), 859–906.
- [5] I. Kachkovskiy and Yu. Safarov, Distance to normal elements in C^* -algebras of real rank zero. *Journal of the American Mathematical Society* **29** (2016), 61–80.
- [6] H. Lapointe, I. Polterovich and Yu. Safarov, Average growth of the spectral function on a Riemannian manifold. *Communications in Partial Differential Equations* **34** (2009), 581–615.
- [7] A. Laptev and Yu. Safarov, Szegő type limit theorems. *Journal of Functional Analysis* **138** (1996), 544–559.
- [8] A. Laptev, Yu. Safarov and D. Vassiliev, On global representation of Lagrangian distributions and solutions of hyperbolic equations *Communications on Pure and Applied Mathematics* **47** (1994), 1411–1456.
- [9] London Mathematical Society, Citation for Yuri Safarov. *Bulletin of the London Mathematical Society* **29** (1997), 384.
- [10] P. McKeag and Yu. Safarov, Pseudodifferential operators on manifolds: a coordinate-free approach. In *Partial Differential Equations and Spectral Theory*, volume 211 of the series *Operator Theory: Advances and Applications* (2011), 321–341.
- [11] Yu. Netrusov and Yu. Safarov, Weyl asymptotic formula for the Laplacian on domains with rough boundaries. *Communications in Mathematical Physics* **253** (2005), 481–509.
- [12] Yu. Safarov, Spectral asymptotics for the Maxwell operator and transmission problems. PhD thesis, Leningrad Branch of the Steklov Mathematical Institute of the USSR Academy of Sciences, 1984. In Russian.
- [13] Yu. Safarov, Non-classical two-term spectral asymptotics for self-adjoint elliptic operators. DSc thesis, Leningrad Branch of the Steklov Mathematical Institute of the USSR Academy of Sciences, 1989. In Russian.
- [14] Yu. Safarov, Pseudodifferential operators and linear connections. *Proceedings of the London Mathematical Society* **74** (1997), 379–416.
- [15] Yu. Safarov, Fourier Tauberian theorems and applications. *Journal of Functional Analysis* **185** (2001), 111–128.

- [16] Yu. Safarov and D. Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators*. Amer. Math. Soc., Providence (RI), 1997 (hardcover), 1998 (softcover).